M 392C, Fall 2025: Dr. David Ben-Zvi's notes, TEXed by Sai Sivakumar.

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Lecture 1 08/26

Definitions

Let G be a group and k be a field. A representation of G is a vector space $V \in \mathbf{Vect}_k$ with a linear action of G on V.

This is to say there is a homomorphism $G \xrightarrow{\rho} \mathrm{Aut}(V)$ so that we may define the action of G on V to be $g \cdot v := \rho(g)v$ or alternatively a product $G \times V \to V$ given by $(g, v) \mapsto \rho(g)v$ that is linear in V and associative in G, among other properties. (We often suppress the $\cdot$ in the action/product.)

Note $\mathrm{Aut}(V) =: \mathrm{GL}(V) \cong \mathrm{GL}_{\dim V}(k)$, so we could even think of the elements of G as matrices if we fix a basis of V (and V is finite dimensional).

The representations (V, ρ) of G over a field k form the objects of a category $\mathbf{Rep}_k(G)$ whose morphisms are linear transformations which commute with the action of G; i.e., $\mathrm{Hom}_G(V, W)$ consists of the linear transformations $T \colon V \to W$ for which $g(Tv) = T(gv)$ for all $g \in G$ and $v \in V$. (These are also called intertwining operators or G-linear maps, etc. Also note that $\mathrm{Hom}_G(V, W)$ is merely a vector space, but see this MSE post.)

Variants of representations

We can add adjectives in various places to the definitions above to get different kinds of representations.

For example, if V has an inner product we can ask about linear actions of G on V that preserve that inner product. If V is a Hilbert space we can ask about linear actions of G on V for which $\rho(g)$ is a unitary operator; these are the unitary representations of G.

If G has a topology and V is a topological vector space, we say (V, ρ) is a continuous representation if the product map $G \times V \to V$ defined before is continuous. If G has a smooth structure (e.g. is a Lie group) or is an algebraic group, we can ask about smooth or algebraic representations, respectively.

Aspects of representations we study

1. Irreducible representations (henceforth called "irreps"): An irrep of G are the representations V which have no invariant (or stable some might say) subspaces under the group action; that is, if W is a G-invariant subspace of V then W is either 0 or V. These are like the "atoms" in representation theory.

   For a given group G one goal of representation theory is to classify up to isomorphism its irreps. One approach is to try to attach numerical invariants to representations and see if they help to classify irreps, and generalizations of this idea lead to character theory. Sometimes there are "isotopes" that are not isomorphic but nevertheless cannot be distinguished by certain invariants.

Another thing we do is to take a subgroup H of G and study how irreps of G decompose when the group action is restricted to H. On the other hand, we can also look at how to build bigger representations or "molecules" out of the atomic irreps via extension. Of course, it is easy to take direct sums of irreps but depending on the context it may be possible to obtain indecomposable reps which are extensions of irreps but do not decompose into a direct sum of irreps (so to summarize, irreducible implies indecomposable but not the other way around in general).

Maschke's theorem implies that indecomposable representations of finite groups over fields with characteristic not dividing the order of the group are irreducible. Alternatively, the theorem implies that in this setting, all short exact sequences of representations split.

For example, the indecomposable representations over $\mathbb{C}$ coincide with the irreps when G is finite. On the other hand, the shearing representation

$$\begin{aligned} \mathbb{Z} &\to \mathrm{GL}_2(\mathbb{R}) \\ 1 &\mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \end{aligned}$$

preserves the horizontal axis, so it is not an irrep, but is not decomposable since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable. The representations (k^n, ρ) of $\mathbb{Z}$ are given by specifying an invertible $n \times n$ matrix, and any two such representations are isomorphic if the matrices specifying them are conjugate. If k is algebraically closed, the indecomposable representations in this setting correspond to Jordan blocks. Of course, every representation is the direct sum of indecomposables, and in this setting the Jordan normal form of an $n \times n$ matrix would describe the decomposition of k^n into indecomposables.

2. Harmonic analysis is the study of naturally appearing "large" representations; we would like to perform some kind of "spectroscopy" to determine what representations appear within them.

   If a group G acts on some object X (which could be a set, manifold, or otherwise a "geometric" object), we say X has some symmetries which we would like to "linearize". We achieve this by considering the k-valued functions on X for some field k; if X has additional structure we can restrict to functions on X which interact with that structure (e.g. measurable/continuous/etc maps)

   From a right action of G on X, function spaces on X inherit a natural linear left action of G; one is given by $gf(x) = f(xg)$. These representations are typically very decomposable or reducible.

3. Different groups give rise to different phenomena. Below are some examples of groups we will see again.

	compact	noncompact
Abelian	S^1	$\mathbb{R}$
non-Abelian	SO(3)	$\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$

Irreps of compact groups coincide with their indecomposables. Irreps of Abelian groups are all 1-dimensional.

The representation theory of S^1 leads to the theory of Fourier series, and in a similar way we can recover the Fourier transform from the representation theory of $\mathbb{R}$.

The Lie group SO(3) can be thought of as the group of rotations of a 2-sphere. One nice result is that the spherical harmonics are basis functions for the irreps of SO(3), which occur naturally as the atomic orbitals (see the Wikipedia article on spherical harmonics).

The representations of the groups $\mathrm{SL}_2(k)$ for various fields k appear all over math. We can study special functions like the Bessel or hypergeometric functions, or even modular forms. The hard Lefschetz theorem in algebraic geometry says that $\mathrm{SL}_2(\mathbb{C})$ acts on $H^*(X, \mathbb{C})$ for X a nice enough smooth projective variety. In physics, the special linear group sort of appears in the Lorentz group $\mathrm{SO}(1,3)^+ = \mathrm{PSL}_2(\mathbb{C})$.

Spectral theory

A slogan for what is to come: "commutativity implies geometry".

Let $k = \mathbb{C}$ and X a set. Then the complex-valued functions on X form a commutative algebra. This is some example of a functor suggestively called $\mathcal{O}$ from the category of some kind of geometric objects to commutative algebras.

To expand on the previous idea, here is a motto originating from Gelfand and Grothendieck's work:

1. Any commutative ring should be thought of as functions on some space.

   That is, there is some functor going from commutative rings to some category of geometric objects that realizes this idea. In particular we should be able to adjust the functor and its source/target to obtain an equivalence of categories.

   $$\text{commutative rings} \longleftrightarrow \text{geometry}$$

2. Once we are in the situation where we have an equivalence of categories between commutative rings and geometric objects, we should further obtain a correspondence between the modules over a ring R and sheaves (special families of vector spaces or Abelian groups) on the corresponding geometric object X to R.

   $$R\text{-modules} \longleftrightarrow \text{sheaves on } X$$

For example, if X is a finite set, the corresponding ring R is the finite-dimensional commutative algebra of complex valued functions on X. This algebra is semisimple with $R = \bigoplus_i \mathbb{C}e_i$ where $e_i e_j = \delta_{ij}$. We can think of the e_i as delta/indicator functions on points of X.

An R-module M has the decomposition $M = \oplus_i e_i M$, which corresponds to a sheaf on X where at each point of

X we imagine the corresponding module e_iM lying on it:

$$\begin{array}{ccccc} M & e_1M & e_2M & \cdots & e_nM \\ X & \bullet & \bullet & \cdots & \bullet \end{array}$$

A short word about the group algebra

A representation V of a group G is given by a group homomorphism $G \xrightarrow{\rho} \mathrm{Aut}(V)$. Since $\mathrm{Aut}(V)$ is contained in $\mathrm{End}(V)$, a k-algebra, there is an object $k[G]$ called the group algebra for which the homomorphism ρ factors through $k[G]$; that is, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{End}(V) \\ \big\downarrow & \overset{\bar{\rho}}{\nearrow} & \\ k[G] & & \end{array}$$

commutes. One explicit description of $k[G]$ is the set of finite linear combinations of elements of G with coefficients in k:

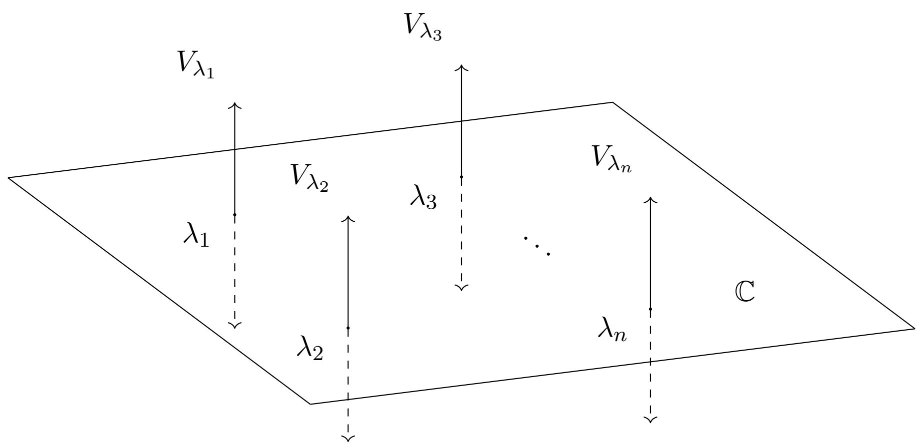
$$k[G] = \left\{ \sum_{g \in G} c_g g \;\middle|\; c_g \in k, \text{ all but finitely many } c_k \text{ are zero} \right\}.$$

The addition and multiplication in $k[G]$ are defined using the addition in k and multiplication in G, respectively. We will return to the group algebra when investigating representations from a module-theoretic point of view.

Lecture 2 08/28

Spectral theory over $\mathbb{C}$

Consider an operator T on a finite dimensional complex vector space V. The spectrum of T, $\sigma(T)$, is a finite subset of $\mathbb{C}$ consisting of the eigenvalues of T. To each $\lambda \in \sigma(T)$ there is a corresponding eigenspace V_λ of V, and $V = \bigoplus_{\lambda\in\sigma(T)} V_\lambda$. We can visualize placing each of the eigenspaces V_λ above each point λ in the spirit of sheaf theory:



On $\mathbb{C}$, the coordinate function x where $x(z) = z$ has the same action as T on the eigenspaces V_λ:

$$Tv = \sum_{\lambda\in\sigma(T)} Tv_\lambda = \sum_{\lambda\in\sigma(T)} \lambda v_\lambda = \sum_{\lambda\in\sigma(T)} x(\lambda)v_\lambda = x\cdot\left(\sum_{\lambda} v_\lambda\right) = x\cdot v$$

In this sense T corresponds to $x \in \mathrm{Fun}(\mathbb{C})|_{\sigma(T)}$ (complex-valued functions on $\mathbb{C}$, restricted to $\sigma(T)$.).

Now consider the commutative ring $R = k[T_1,\dots,T_n]/(f_1,\dots,f_m)$ and an R-module M. We would like to simultaneously diagonalize the action of the T_i on M, or rather, diagonalize the action of R on M, which amounts to finding a basis of M where R acts diagonally. Assume we can do this.

We should attach a set $X = \mathrm{Spec}(R)$ to R called the spectrum of R for which we can decompose M into a direct sum of modules M_{x_i}, each summand lying over their corresponding point $x_i \in X$:

$$\begin{array}{ccccccccc} M &=& M_{x_1} &\oplus& M_{x_2} &\oplus\cdots\oplus& M_{x_\ell} \\ X && x_1 && x_2 & \cdots & x_\ell \end{array}$$

Furthermore, we form an assignment $R \xrightarrow{\varphi} \mathrm{Fun}(X)$ for which we can recast the action of R on M through this assignment:

$$rm = \varphi(r)\sum_{x\in X} m_x = \sum_{x\in X} \varphi(r)|_{\{x\}} m_x$$

The idea here is to turn what was an algebraic notion of rings acting on modules to thinking about sheaves on a particular space which "diagonalize" the ring action.

Some examples of the algebra-geometry dictionary

An important philosophy we've been looking at the broad strokes of is this dictionary between algebra and geometry, specifically the two following ideas coming from Gelfand and Grothendieck:

1. Commutative rings correspond to geometric objects via functors

$$\text{commutative rings} \underset{\mathcal{O}}{\overset{\text{Spec}}{\rightleftarrows}} \text{geometric objects}$$

   In this vague setting we should think of Spec as in taking the spectrum of some collection of simultaneously diagonalized operators coming from the ring and $\mathcal{O}$ as returning the space of functions on these geometric objects. Again we should think of R as being the space of functions on $\operatorname{Spec} R$ in this correspondence.

2. Modules over rings R correspond to sheaves on the corresponding geometric object $\operatorname{Spec} R$ via

$$\text{modules} \underset{\text{global sections}}{\overset{\text{spectral decomposition}}{\rightleftarrows}} \text{sheaves}$$

   In particular in the previous examples we have that taking global sections amounts to taking the direct sum of modules, but this can also appear as a direct integral of modules in continuous versions of the previous examples. Spectral decomposition as we have seen is to break up a module into submodules where the ring action is pointwise multiplication.

We look at some examples of part 1. of the above philosophy.

Grothendieck's version of this idea is the heart of modern algebraic geometry. One correspondence is

$$\text{commutative rings} \longleftrightarrow \text{affine schemes}$$

but an earlier version might have been

$$\text{finitely presented, reduced, etc. } \mathbb{C}\text{-algebras} \longleftrightarrow \text{complex affine varieties}$$

In both correspondences, the Spec functor is given by taking the set of prime ideals. In the top correspondence, $\mathcal{O}$ is taking the structure sheaf of a scheme, but this amounts to taking polynomial functions on a space in the bottom correspondence.

Gelfand's version of this idea is the correspondence

$$\text{commutative } C^*\text{-algebras} \longleftrightarrow \text{Hausdorff locally compact topological spaces}$$

The functor Spec in this case is the eponymous Gelfand spectrum and $\mathcal{O}$ takes the continuous compactly supported functions on Hausdorff, locally compact spaces.

A special case of the above is the correspondence

$$\text{commutative von Neumann algebras} \longleftrightarrow \text{measure spaces}$$

One direction is some kind of spectrum, but the other direction is taking L^∞ functions on a measure space. As a side remark, there are only five von Neumann algebras up to equivalence; they are: finite sets, $\mathbb{N}$, $[0,1]$, $[0,1]$ union a finite set, and $[0,1]$ union a countably infinite set.

There is a version of part 2. for each of the above examples involving modules and sheaves, but we will not discuss them here, aside from mentioning that we can talk about algebraic, continuous, or measurable families of vector spaces (sheaves) in the various settings above.

The group algebra (over $\mathbb{C}$)

Let G be any group and (V, ρ) any complex representation of G (the below discussion would work for other fields instead of $\mathbb{C}$). Recall that $\mathbb{C}[G]$ is the unique object for which the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\rho} & \mathrm{Aut}(V) & \hookrightarrow & \mathrm{End}(V) \\ \downarrow & & & \nearrow_{\bar{\rho}} & \\ \mathbb{C}[G] & & & & \end{array}$$

commutes; moreover, we think of $\mathbb{C}[G]$ as a "free" $\mathbb{C}$-algebra on G. This is because the functor taking G to $\mathbb{C}[G]$ is the left adjoint to the forgetful functor from $\mathbb{C}$-**alg** to **Group** (given by taking the group of units).

An explicit description of $\mathbb{C}[G]$ is the $\mathbb{C}$-vector space generated by the elements of G with multiplication induced by the products in $\mathbb{C}, G$. An alternative description of $\mathbb{C}[G]$ is the set of finitely supported functions from G to $\mathbb{C}$, with multiplication given by convolution:

$$(f * h)(g) = \sum_{g=xy} f(x)h(y) \quad \text{(defined since } f, h \text{ are finitely supported)}$$

This product is no different from the product defined in the first description of $\mathbb{C}[G]$.

What is really happening is that we can take f, h from above and form the box product $f \boxtimes h \colon G \times G \to \mathbb{C} \times \mathbb{C}$, which is just the map $(x, y) \mapsto (f(x), h(y))$. We want to form a map from $G \times G$ to $\mathbb{C}$, so consider $G \times G \xrightarrow{m} G$, the product in G. The pushforward of $f \boxtimes h$ by m is $f * h$:

$$m_*(f \boxtimes h) \quad \text{also denoted} \quad \int_m f \boxtimes h = \sum_{-=xy} f(x)h(y) = f * h$$

An important observation to make is that complex representations of G coincide with $\mathbb{C}[G]$-modules.

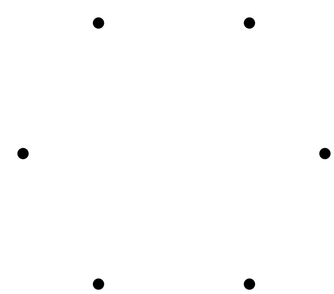
Representation theory of finite Abelian groups and the dual group

If G is Abelian, then $\mathbb{C}[G]$ is a commutative ring. So representation theory may be thought of as a special case of the study of modules over commutative rings. Using language from earlier, note that the corresponding geometric object to $\mathbb{C}[G]$ is $\operatorname{Spec}(\mathbb{C}[G])$, and representations of G correspond to vector bundles or sheaves over $\operatorname{Spec}(\mathbb{C}[G])$.

Let G be a finite Abelian group, and let $\widehat{G} \coloneqq \operatorname{Spec}(\mathbb{C}[G])$; we call this the dual of G. Then $\mathbb{C}[G]$ (functions on G) with multiplication given by convolution is isomorphic as an algebra to $\mathbb{C}[\widehat{G}]$ (functions on $\widehat{G}$, which we will see is a finite set) with pointwise multiplication. This isomorphism is usually given by some kind of finite Fourier transform. On the other hand, representations of G correspond to vector bundles (sheaves) on $\widehat{G}$, and this is some kind of finite spectral theorem.

Let $G = \langle x \rangle$ be a cyclic group of order n. The dual object $\widehat{G}$ is given by the n-th roots of unity in $\mathbb{C}$: (the picture is for $n = 6$)

$\widehat{G} = \operatorname{Spec}(\mathbb{C}[G]) =$



In this case, Spec is taking the maximal ideals of rings. With $\mathbb{C}[G] \cong \mathbb{C}[x]/(x^n - 1)$, it is equivalent to describe the elements in $\operatorname{Hom}_{\mathbb{C}\textbf{-alg}}(\mathbb{C}[x]/(x^n-1), \mathbb{C})$, which are characterized by each of the n-th roots of unity.

We take a small detour to discuss Schur's lemma, which states that for any group G, that any $\mathbb{C}[G]$-module homomorphisms between simple $\mathbb{C}[G]$-modules are either 0 or are isomorphisms of $\mathbb{C}[G]$-modules. This follows by investigating kernels and images since they are submodules of V, W respectively. In particular, any $\mathbb{C}[G]$-module endomorphism of a simple $\mathbb{C}[G]$-module is a scalar multiple of the identity, since for any nonzero endomorphism T we can consider $T - \lambda \operatorname{id}_V$, which is no longer an isomorphism and hence must be zero. Here we used the algebraic closedness of $\mathbb{C}$, and in fact we could have replaced $\mathbb{C}$ by any algebraically closed field k.

Schur's lemma is used to prove that if G is Abelian then any finite-dimensional irrep of G is one-dimensional. If such an irrep V had finite dimension greater than or equal to 2, then left multiplication $V \xrightarrow{g} V$ by any element $g \in G$ must be a scalar multiple of id_V, which contradicts the irreducibility of V (since we assumed $\dim V \geq 2$).

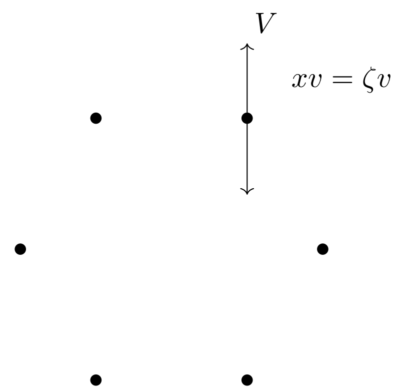
The action of G on the irrep V in this case is given by scalar multiplication by $\chi(g) \in \mathbb{C}^*$ since V is one-dimensional. More importantly, the assignment $g \to \chi(g)$ is a group homomorphism $G \to C^*$, which we call a character of G.

For G Abelian, $\widehat{G}$ is the set of maximal ideals of $\mathbb{C}[G]$, which is in bijection with $\operatorname{Hom}_{\mathbb{C}\textbf{-alg}}(\mathbb{C}[G], \mathbb{C})$. By the universal property of the group algebra, $\mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{alg}}(\mathbb{C}[G], \mathbb{C})$ is in bijection with $\mathrm{Hom}_{\mathbf{Group}}(G, \mathbb{C}^*)$:

$$\begin{array}{ccccc} G & \xrightarrow{f} & \mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^* & \hookrightarrow & \mathrm{End}(\mathbb{C}) \cong \mathbb{C} \\ \big\downarrow & & & \overset{\overline{f}}{\nearrow} & \\ \mathbb{C}[G] & & & & \end{array}$$

So in general we can also think of $\widehat{G}$ as the collection of irreps of G.

Given an irrep V of G (i.e. a simple $\mathbb{C}[G]$-module), we can try to form a sheaf out of V on $\widehat{G}$. Since V is irreducible, this sheaf has to be concentrated at only one of the points of $\widehat{G}$, and this point is the point corresponding to the irrep V itself. So in the example where $G = \langle x \rangle$ has order n, the point at which an irrep V lies on is the root of unity ζ for which the action of x on an irrep V is given by multiplication by ζ.



Since $\widehat{G}$ is in bijection with $\mathrm{Hom}_{\mathbf{Group}}(G, \mathbb{C}^*)$, we can equip $\widehat{G}$ with a group operation; by doing so, G and $\widehat{G}$ are non-canonically isomorphic as groups. So in particular finite cyclic groups are (non-canonically) self-dual.

Next time we will explore the Fourier transform, which is a $\mathbb{C}[G]$-module isomorphism

$$\mathbb{C}[G] \xrightarrow{\widehat{\ }} \mathbb{C}[\widehat{G}]$$

where $\mathbb{C}[G]$, $\mathbb{C}[\widehat{G}]$ are thought of as function spaces and are given convolution and pointwise multiplication, respectively (we will describe the action of G on each of these spaces next time). This isomorphism has some symmetry which is part of the statement of Pontryagin duality.

Lecture 3 09/02

Lecture 4 09/04

References