These notes follow *Modular Forms* by Toshitune Miyake and *A First Course in Modular Forms* by Fred Diamond and Jerry Shurman.

1 Conventions and notations

The upper half plane of $\mathbb{C}$ is denoted $\mathbb{H}$. By adding additional points and charts, obtain the Riemann surface $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. These extra points contain the cusps of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In the analysis below, we ignore cusps in order to streamline this note. For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the corresponding compactified modular curve is the Riemann surface $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the factor of automorphy is $j(\gamma, z) = cz + d$ for $z \in \mathbb{C}$.

A modular form of weight k with respect to a congruence subgroup Γ is a holomorphic function f on $\mathbb{H}$ for which

$$f(\gamma(z)) = j(\gamma, z)^k f(z)$$

for any $\gamma \in G$ and $z \in \mathbb{H}$. Furthermore, we require that $j(\alpha, z)^{-k} f(\alpha(z))$ is holomorphic at ∞ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ (meaning the Fourier coefficients of this function are concentrated in indices greater than or equal to zero). This last condition captures the condition that f be holomorphic at the cusps for Γ.

An automorphic form (of weight k) is a meromorphic function f on $\mathbb{H}$ which is meromorphic at the cusps of Γ and transforms with respect to a congruence subgroup in the same way as in the definition of a modular form. That is, for any $\gamma \in \Gamma$ and $z \in \mathbb{H}$ we have $f(\gamma(z)) = j(\gamma, z)^k f(z)$.

Let z be a coordinate of $U \subseteq \mathbb{C}$. The meromorphic differentials on U are meromorphic (global) sections of the cotangent bundle Ω of U, and a meromorphic section of Ω is a (holomorphic) section $\omega = f(z)\mathrm{d}z \in \Omega(V)$ for some open set $V \subseteq U$ where $f(z)$ is meromorphic on U and $U \setminus V$ contains the poles of $f(z)$. In other words, the meromorphic differentials on U are sections of the form $f(z)\mathrm{d}z$ where $f(z)$ is meromorphic on U.

The meromorphic differentials of degree n on U are meromorphic (global) sections of the n-th tensor (or symmetric) power of the cotangent bundle of U, $\Omega^{\otimes n}$. Since Ω is a trivial line bundle, its tensor powers are also trivial, so sections of $\Omega^{\otimes n}$ are of the form $f(z)(\mathrm{d}z)^{\otimes n}$. Since the cotangent bundle is rank one, its tensor powers agree with its symmetric powers, so we suppress the notation $\otimes$ from now on. Denote meromorphic differentials of degree n on U by $f(z)(\mathrm{d}z)^n$ for $f(z)$ meromorphic on U and the space of meromorphic differentials of degree n on U by $\Omega^n(U)$. There is a natural way to multiply differentials (of possibly different degrees) together, so these spaces assemble into a graded algebra of differentials $\Omega^\bullet(U)$.

If $\varphi\colon U \to V$ is a holomorphic map of open sets in $\mathbb{C}$, then the pullback map $\varphi^*\colon \Omega^n(V) \to \Omega^n(U)$ is given by the change of variables $w = \varphi(z)$:

$$f(w)(\mathrm{d}w)^n \mapsto f(\varphi(z))(\varphi'(z))^n(\mathrm{d}z)^n,$$

where w is the coordinate for V and z is the coordinate for U.

Let R be a compact Riemann surface. A meromorphic differential of degree k on R is given by an atlas $\{(V_\alpha, V_\alpha \xrightarrow{\varphi_\alpha} U_\alpha)\}$ of R and a collection of meromorphic functions $\{\phi_\alpha\}$ on R for which

$$\phi_\alpha(p)\left(\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}\varphi_\beta}(p)\right)^k = \phi_\beta(p)$$

for all $p \in V_\alpha \cap V_\beta$. This condition is more accurately/less confusingly(?) presented as the following condition: The functions ϕ_α in local coordinates determine the meromorphic differentials $(\phi_\alpha \circ \varphi_\alpha^{-1})(z)(\mathrm{d}z)^k$ on U_α for each α, and we require that the pullback of a differential $(\phi_\beta \circ \varphi_\beta^{-1})(z)(\mathrm{d}z)^k$ along the transition map $U_\alpha \xrightarrow{\varphi_\beta\varphi_\alpha^{-1}} U_\beta$ agrees with $(\phi_\alpha \circ \varphi_\alpha^{-1})(z)(\mathrm{d}z)^k$ on $U_\alpha \cap U_\beta$; that is,

$$\begin{aligned}(\varphi_\beta\varphi_\alpha^{-1})^*((\phi_\beta \circ \varphi_\beta^{-1})(z)(\mathrm{d}z)^k) &= (\phi_\beta \circ \varphi_\beta^{-1})(\varphi_\beta\varphi_\alpha^{-1}(w))\left(\frac{\mathrm{d}\varphi_\beta\varphi_\alpha^{-1}(w)}{\mathrm{d}w}\right)^k(\mathrm{d}w)^k \\ &= (\phi_\beta \circ \varphi_\alpha^{-1})(w)\left(\frac{\mathrm{d}\varphi_\beta\varphi_\alpha^{-1}(w)}{\mathrm{d}w}\right)^k(\mathrm{d}w)^k\end{aligned}$$

agrees with

$$(\phi_\alpha \circ \varphi_\alpha^{-1})(w)(\mathrm{d}w)^k$$

on $U_\alpha \cap U_\beta$ (and the same can be said about pulling back along the transition function in the other direction). There is a notion of equivalence of meromorphic differentials which we do not record here. Denote the space of meromorphic differentials of degree n on R by $\Omega^n(R)$ and note that these spaces also assemble into the graded algebra $\Omega^\bullet(R)$.

The order of vanishing of a nonzero meromorphic differential $\omega = \{\phi_\alpha\}$ of degree k of R at the point $p \in V_\alpha$, denoted $v_p(\omega)$ is given by the usual order of vanishing of ϕ_α at p, calculated in local coordinates by $v_{\varphi_\alpha(p)}(\phi_\alpha \circ \varphi_\alpha^{-1})$. This value is independent of the local representative used for ω because the transition functions are holomorphic and have no zeroes on the intersection of two charts. Therefore the definition of the divisor of ω given by

$$\mathrm{div}(\omega) = \sum_p v_p(\omega)p \in \mathrm{Div}(R) = \mathbb{Z}R$$

makes sense for nonzero differentials. This definition of the divisor function is additive on products of differentials as expected. The degree of a divisor $a = \sum_p a_p p$ is $\deg(a) = \sum_p a_p \in \mathbb{Z}$.

A divisor D is nonnegative if all of its coefficients are nonnegative, and denote this by $D \geq 0$. The Riemann-Roch space of a divisor D is the vector space

$$L(D) = \{f \in \mathbb{C}(R)^\times \mid f = 0 \text{ or } \mathrm{div}(f) + D \geq 0\}$$

and its dimension is denoted $l(D)$.

The Riemann-Roch theorem: Let g be the genus of R and choose any nonzero differential ω of degree 1 on R (we call $\mathrm{div}(\omega)$ a canonical divisor on R). Then for any divisor a,

$$l(a) = \deg(a) - (g-1) + l(\mathrm{div}(\omega) - a).$$

Note that $l(0) = 1$ since only the constant functions in $\mathbb{C}(R)$ have divisor greater than or equal to zero (by compactness of R, the only holomorphic functions on R are the constants). So with $a = 0$, deduce that for any canonical divisor $\mathrm{div}(\omega)$ on R that $l(\mathrm{div}(\omega)) = g$. With $a = \mathrm{div}(\omega)$, deduce that $\deg(\mathrm{div}(\omega)) = 2(g-1)$. If $l(a) > 0$, then we can select a nonzero $f \in L(a)$ from which we see that $\mathrm{div}(f) + a \geq 0$; taking the degree yields $\deg(\mathrm{div}(f)) + \deg(a) \geq 0$, so $\deg(a) \geq -\deg(\mathrm{div}(f)) = 0$ (the degree of divisors of meromorphic functions on a compact Riemann surface, called principal divisors, is always zero). By the contrapositive, if $\deg(a) < 0$, then $l(a) = 0$. This implies that if $\deg(a) > 2(g-1)$; that is, for a canonical divisor $\mathrm{div}(\omega)$ that $\deg(\mathrm{div}(\omega) - a) < 0$, then $l(\mathrm{div}(\omega) - a) = 0$.

2 Meromorphic differentials and modular forms

Since most of $\mathbb{H}^*$ is $\mathbb{H}$, we will refer to meromorphic differentials $\{\phi_\alpha\}$ of degree k on $\mathbb{H}^*$ simply by the meromorphic ϕ on $\mathbb{H}$. Since the coordinate on $\mathbb{H}$ is given by z (the chart map is the identity), ϕ determines a meromorphic differential $\phi(z)(\mathrm{d}z)^k$ on $\mathbb{H} \subseteq \mathbb{C}$. We will identify meromorphic differentials on $\mathbb{H}^*$ with meromorphic differentials on $\mathbb{H}$ in this way.

An automorphic form $f(z)$ on $\mathbb{H}$ can be sent to the meromorphic differential $f(z)(\mathrm{d}z)^n$ on $\mathbb{H}^*$. This differential form is Γ-invariant: for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\mathrm{d}(\gamma(z)) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{az+b}{cz+d}\right)\mathrm{d}z = \frac{1}{(cz+d)^2}\mathrm{d}z = j(\gamma,z)^{-2}\mathrm{d}z$$

(since $\det\gamma = 1$). Thus $f(\gamma(z))(\mathrm{d}(\gamma(z)))^k = j(\gamma,z)^{-2k}f(z)j(\gamma,z)^{2k}(\mathrm{d}z)^k = f(z)(\mathrm{d}z)^k$. In other words, this gives a map

$$\{\text{automorphic forms } f(z) \text{ of weight } 2k\} \xrightarrow{(-)(\mathrm{d}z)^k} \{\Gamma\text{-invariant meromorphic differentials } f(z)(\mathrm{d}z)^k \text{ on } \mathbb{H}^*\}.$$

By reversing the calculation above, it follows that this map is an isomorphism of vector spaces.

A meromorphic differential on $X(\Gamma)$ may be pulled back along the quotient map $\pi\colon \mathbb{H}^* \to X$ to obtain a meromorphic differential on $\mathbb{H}^*$. To understand this pullback explicitly, start with a known set of charts V_i for $X(\Gamma)$ and pull back each $\phi_i(z)(\mathrm{d}z)^n$ on each one of those charts. We will not record the calculations for the pullback maps using the usual charts for $X(\Gamma)$, but the point is that because a meromorphic differential on $X(\Gamma)$ is specified by a collection of meromorphic differentials on open sets which glue together, these meromorphic differentials will still glue together after being pulled back to meromorphic differentials on $\mathbb{H}^*$.

Since π is the quotient given by identifying orbits of the action of Γ on $\mathbb{H}^*$ to points, a meromorphic differential $f(z)(\mathrm{d}z)^n$ on $\mathbb{H}^*$ obtained by pulling back a meromorphic differential on $X(\Gamma)$ along the quotient map π is Γ-invariant. So for any $\gamma \in \Gamma$, we can pull back $f(z)(\mathrm{d}z)^n$ along the action of γ on $\mathbb{H}^*$ to obtain equalities

$$f(z)(\mathrm{d}z)^n = \gamma^*(f(z)(\mathrm{d}z)^n) = f(\gamma(z))(\gamma'(z))^n(\mathrm{d}z)^n = j(\gamma,z)^{-2n}f(\gamma(z))(\mathrm{d}z)^n.$$

This shows that the function $f(z)$ is an automorphic form with weight $2n$. To see that $f(z)$ is meromorphic at cusps, it suffices to see that the meromorphic differential it comes from expanded locally around the cusps of $X(\Gamma)$ is given by a function which is meromorphic at zero, so $f(z)$ should also be meromorphic at its cusps.

The above discussion establishes the following map

$$\{\text{meromorphic differentials on } X(\Gamma)\} \xrightarrow{\pi^*} \{\Gamma\text{-invariant meromorphic differentials } f(z)(\mathrm{d}z)^k \text{ on } \mathbb{H}^*\}$$
$$\cong \{\text{automorphic forms } f(z) \text{ of weight } 2k\}.$$

That π^* is an isomorphism is not difficult but is tedious to show (this amounts to Theorem 2.3.1 in Miyake, which proves that the composite of the maps above is an isomorphism). The calculation one must do is to show that a weakly meromorphic function $f(z)$ of weight $2k$ gives rise to a meromorphic differential on $X(\Gamma)$ of degree k in a way which is (left and right) inverse to the composite map above.

We will do part of the calculation. Let $f(z)$ be an automorphic form of weight $2k$ on $\mathbb{H}^*$. We will describe the component of the meromorphic differential $\omega = \{\phi_\alpha\}$ of degree k it defines on $X(\Gamma)$ around a point $a = \pi(z_0) \in X(\Gamma)$ for $z_0 \in \mathbb{H}$. That is, we will define a meromorphic function ϕ_a that ω looks like around a. There is a neighborhood U_{z_0} of z_0 in $\mathbb{H}$ such that $\gamma U_{z_0} \cap U_{z_0} \neq \emptyset$ if and only if $\gamma \in \Gamma_{z_0}$, in which case $\gamma U_{z_0} = U_{z_0}$, and a chart $(V_\alpha = \pi(U_{z_0}), V_a \xrightarrow{\varphi_a} U_a)$ containing a. Consider the function

$$g(z) = f(z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(z)}{\mathrm{d}z}\right)^{-k}$$

defined on U_{z_0}. In principle we could define $g(z)$ on all Γ-translates of U_{z_0}, but since π is the quotient map $\mathbb{H}^* \to \Gamma\backslash\mathbb{H}^*$, it is enough to look at U_{z_0}. We will show that $g(z)$ descends to a meromorphic function on V_a; that is, there exists a function ϕ_a on V_a such that $(\phi_a \circ \pi)(z) = g(z)$ on U_{z_0}. This amounts to checking that $g(z)$ is Γ_{z_0}-invariant (we don't check that $g(z)$ is Γ-invariant since the domain of $g(z)$ is U_{z_0} and the automorphisms of that set are Γ_{z_0}). Indeed, for any $\gamma \in \Gamma_{z_0} \subset \Gamma$, $\pi(\gamma(z)) = \pi(z)$ (as π sends z to Γz) so

$$\begin{aligned}
g(\gamma(z)) &= f(\gamma z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(\gamma z)}{\mathrm{d}\gamma z}\right)^{-k} \\
&= j(\gamma, z)^{2k} f(z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(z)}{\mathrm{d}z}\right)^{-k}\left(\frac{\mathrm{d}(\gamma z)}{\mathrm{d}z}\right)^{-k} \\
&= j(\gamma, z)^{2k} f(z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(z)}{\mathrm{d}z}\right)^{-k} j(\gamma, z)^{-2k} \\
&= f(z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(z)}{\mathrm{d}z}\right)^{-k} = g(z)
\end{aligned}$$

Thus $g(z)$ descends to a meromorphic function ϕ_a on V_a, which is what we sought to calculate.

In summary, the three vector spaces below are isomorphic:

1. the space of automorphic forms $f(z)$ of weight $2k$
2. the space of Γ-invariant meromorphic differentials $f(z)(\mathrm{d}z)^k$ on $\mathbb{H}^*$
3. the space of meromorphic differentials on $X(\Gamma)$ of degree k.

The weight zero automorphic forms correspond to the degree 0 meromorphic differentials on $X(\Gamma)$, in other words, the meromorphic functions on $X(\Gamma)$. Replacing meromorphic with holomorphic in the above list gives the correspondence of modular forms of weight $2k$ with Γ-invariant holomorphic differentials $f(z)(\mathrm{d}z)^k$ on $\mathbb{H}^*$ and with holomorphic differentials on $X(\Gamma)$ of degree k. We will give a fourth characterization using divisors in the next section.

3 Riemann-Roch and dimension formulas

Due to the possibility of elliptic points in $X(\Gamma)$, meromorphic modular functions of positive weight $2k$ may have orders of vanishing that are fractional and not just integers. We will make sense of this shortly. To start we will define the $\mathbb{Q}$-vector space of divisors $\mathrm{Div}_{\mathbb{Q}}(X(\Gamma)) = \mathrm{Div}(X(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}$ on $X(\Gamma)$ and redefine the divisor function div to map to $\mathrm{Div}_{\mathbb{Q}}(X(\Gamma))$. Our goal is to carefully reduce calculations involving divisors with rational coefficients to ones involving integer coefficients so that we can use the Riemann-Roch theorem. Henceforth we also suppress the argument (z) of functions.

Let f be any nonzero automorphic form of weight $2k$ (that these exist for all k is a theorem). First we make some observations. The quotient of two automorphic forms of the same weight is weight zero, so any automorphic form of weight $2k$ may be obtained by multiplying f by a suitable automorphic form of weight zero. The weight zero automorphic forms on $\mathbb{H}$ may be identified with $\mathbb{C}(X(\Gamma))$, the meromorphic functions on $X(\Gamma)$.

Let f be any nonzero automorphic form of weight 2. Let ω be the corresponding degree 1 differential on $X(\Gamma)$; observe that $\mathrm{div}(\omega)$ is a canonical divisor on $X(\Gamma)$. Then ω^k is a degree k differential on $X(\Gamma)$ so

$$\deg(\mathrm{div}(\omega^k)) = \deg(k\mathrm{div}(\omega)) = 2k(g-1).$$

On the other hand, since the space of automorphic forms of weight $2k$ is equal to f^k (which has weight $2k$) times the space of automorphic forms of weight zero, it follows by passing to differentials on $X(\Gamma)$ that $\Omega^k(X(\Gamma)) = \mathbb{C}(X(\Gamma))\omega^k$. So for any nonzero $\tilde{\omega} \in \Omega^k(X(\Gamma))$,

$$\deg(\mathrm{div}(\tilde{\omega})) = \deg(\mathrm{div}(\omega_0\omega^k)) = 2k(g-1)$$

for some $\omega_0 \in \mathbb{C}(X(\Gamma))$ (again since principal divisors on compact Riemann surfaces have degree zero). Therefore every nonzero differential in $\Omega^k(X(\Gamma))$ has divisor with degree $2k(g-1)$.

Let f be a nonzero automorphic form of weight $2k$. A modular form is an automorphic form on $\mathbb{H}^*$ which is holomorphic on $\mathbb{H}$ and at the cusps of Γ. So the space of modular forms of weight $2k$ may be identified with the space

$$\{gf \mid g \text{ is an automorphic form of weight zero}\}.$$

Automorphic forms of weight zero pass to meromorphic functions on $X(\Gamma)$, so we may identify these collections of functions together. Automorphic forms of positive weight $2k$ do not pass to meromorphic functions on $X(\Gamma)$ due to how they transform with respect to Γ, they instead pass to meromorphic differentials of degree k as we saw before. What we would like to do is to nevertheless pass to $X(\Gamma)$ and make the following claim: "The space of modular forms of weight $2k$ may be identified with the space

$$\{g \in \mathbb{C}(X(\Gamma)) \mid g = 0 \text{ or } \mathrm{div}(g) + \mathrm{div}(f) \geq 0\}."$$

This sentence does not make sense since $\mathrm{div}(f)$ is not defined. We will extend the definition of $\mathrm{div}\colon \Omega^\bullet(X(\Gamma)) \to \mathrm{Div}(X(\Gamma))$ to a function whose codomain is $\mathrm{Div}_{\mathbb{Q}}(X(\Gamma))$.

Again consider $a = \pi(z_0) \in X(\Gamma)$ for $z_0 \in \mathbb{H}$ and a neighborhood U_{z_0} of z_0 in $\mathbb{H}$ such that $\gamma U_{z_0} \cap U_{z_0} \neq \emptyset$ if and only if $\gamma \in \Gamma_{z_0}$, in which case $\gamma U_{z_0} = U_{z_0}$, and a chart $(V_\alpha = \pi(U_{z_0}), V_a \xrightarrow{\varphi_a} U_a)$ containing a. There exists $\rho \in \mathrm{SL}_2(\mathbb{Z})$ so that $\rho\mathbb{H}$ is the open unit disk in $\mathbb{C}$ and $\rho z_0 = 0$; that is, ρ is the Cayley transform $\rho z = \frac{z - z_0}{z + z_0}$. We can choose the chart around a so that $(\varphi_a \circ \pi)(z) = (\rho z)^e$ for $z \in U_{z_0}$, where e is the order of $\overline{\Gamma}_{z_0}$ in $\mathrm{PSL}_2(\mathbb{Z})$ (this is called the ramification index at a). Let $\omega = \{\phi_\alpha\}$ be the differential that f is sent to, and ϕ_a the local expression of ω around a on V_a. Thus

$$\begin{aligned}(\phi_a \circ \pi)(z) = g(z) &= f(z)\left(\frac{\mathrm{d}(\varphi_a \circ \pi)(z)}{\mathrm{d}z}\right)^{-k} \\ &= f(z)\left(\frac{\mathrm{d}(\rho z)^e}{\mathrm{d}z}\right)^{-k} \\ &= f(z)\left(\frac{\mathrm{d}(\rho z)}{\mathrm{d}z}\right)^{-k}(e(\rho z)^{e-1})^{-k}.\end{aligned}$$

Set $w = \rho z$ so that

$$(\phi_a \circ \pi \circ \rho^{-1})(w) = e^{-k} f(\rho^{-1} w)\left(\frac{\mathrm{d}\rho^{-1} w}{\mathrm{d}w}\right)^k w^{-k(e-1)}.$$

Since $(\varphi_a \circ \pi \circ \rho^{-1})(w) = w^e = \varphi_a(p)$ for $p = \pi(z)$, the order of vanishing at w of the left hand side is

$$v_w(\phi_a \circ \pi \circ \rho^{-1}) = v_w((\phi_a \circ \varphi_a^{-1}) \circ (\varphi_a \circ \pi \circ \rho^{-1})) = e v_{\varphi_a(p)}(\phi_a \circ \varphi_a^{-1}) = e v_p(\phi_a).$$

The order of vanishing at $w = \rho z$ of the right hand side is

$$v_{\rho z}\left((f \circ \rho^{-1})(-)\left(\frac{\mathrm{d}\rho^{-1}(-)}{\mathrm{d}(-)}\right)^k\right) - k(e-1) = v_z(f) - k(e-1)$$

since $\frac{\mathrm{d}\rho^{-1} w}{\mathrm{d}w}$ (equivalently $\frac{\mathrm{d}\rho z}{\mathrm{d}z}$) has neither zeroes nor poles around ρz_0 (z_0). We calculate $v_z(f)$ by expanding f in a Laurent series around z_0 and calculate the order of vanishing at z (the smallest index with nonzero coefficient in the series expansion). Assembling the above calculations and remembering $\omega = \{\phi_\alpha\}$ yields

$$v_p(\omega) = \frac{1}{e} v_z(f) - k\left(1 - \frac{1}{e}\right)$$

so that we should define

$$v_p(f) \coloneqq \frac{1}{e} v_z(f)$$

where $p = \pi(z)$. The above formula even works when f has weight zero. This would allow us to define $\mathrm{div}(f)$ as the formal sum $\sum_p v_p(f)$ as desired. This extended definition of div does all the right things a function called div ought to do. In particular, we have

$$\mathrm{div}(\omega) = \mathrm{div}(f) - k\sum_{\substack{p \\ \text{period } 2}} \frac{1}{2} p - k\sum_{\substack{p \\ \text{period } 3}} \frac{2}{3} p - k\sum_{\substack{p \\ \text{cusp of } \Gamma}} p.$$

We now return to the calculation for the dimensions of spaces of modular forms. Fix a nonzero automorphic form f of weight $2k$. The space of modular forms of weight $2k$ may be identified with the space

$$\{g \text{ automorphic form of weight zero} \mid g = 0 \text{ or } \mathrm{div}(gf) \geq 0\}$$

which we identify with

$$\{g \in \mathbb{C}(X(\Gamma)) \mid g = 0 \text{ or } \operatorname{div}(g) + \operatorname{div}(f) \geq 0\}$$

since f is nonzero. Since g is meromorphic on $X(\Gamma)$, $\operatorname{div}(g)$ is integral, so the condition $\operatorname{div}(g) + \operatorname{div}(f) \geq 0$ is equivalent to the condition $\operatorname{div}(g) + \lfloor \operatorname{div}(f) \rfloor \geq 0$ (where the floor function on divisors means to apply it to the coefficients). It follows that $M_{2k}(\Gamma)$ is isomorphic to $L(\lfloor \operatorname{div}(f) \rfloor) = \{g \in \mathbb{C}(X(\Gamma)) \mid g = 0 \text{ or } \operatorname{div}(g) + \lfloor \operatorname{div}(f) \rfloor \geq 0\}$, so their dimensions are the same; that is, $\dim(M_{2k}(\Gamma)) = l(\lfloor \operatorname{div}(f) \rfloor)$.

From

$$\operatorname{div}(\omega) = \operatorname{div}(f) - k \sum_{\substack{p \\ \text{period } 2}} \frac{1}{2} p - k \sum_{\substack{p \\ \text{period } 3}} \frac{2}{3} p - k \sum_{\substack{p \\ \text{cusp of } \Gamma}} p$$

we obtain the equation

$$\deg(\lfloor \operatorname{div}(f) \rfloor) = \deg(\operatorname{div}(\omega)) + \lfloor (k/2) \rfloor \varepsilon_2 + \lfloor (2k/3) \rfloor \varepsilon_3 + k\varepsilon_\infty,$$

where $\varepsilon_2, \varepsilon_3, \varepsilon_\infty$ are the number of period 2, 3 elliptic points and cusps respectively. Note further that since $\operatorname{div}(\omega)$ is a k power of a canonical divisor, its degree is $2k(g-1)$ with g the genus of $X(\Gamma)$. Analysis of the right hand side of the above equation yields $\deg(\lfloor \operatorname{div}(f) \rfloor) > 2g - 2$ when $k \geq 1$.

The Riemann-Roch theorem simplifies to the following expression since the degree of $\deg(\lfloor \operatorname{div}(f) \rfloor)$ is greater than $2g - 2$:

$$l(\lfloor \operatorname{div}(f) \rfloor) = (2k-1)(g-1) + \lfloor (k/2) \rfloor \varepsilon_2 + \lfloor (2k/3) \rfloor \varepsilon_3 + k\varepsilon_\infty;$$

this expression is hence equal to the dimension of $M_{2k}(\Gamma)$.

To calculate the dimensions of spaces of cusp forms $S_{2k}(\Gamma)$, observe that since cusp forms vanish at cusps, then $S_{2k}(\Gamma) \cong \{g \in \mathbb{C}(X(\Gamma)) \mid \operatorname{div}(g) + \lfloor \operatorname{div}(f) - \sum_{\text{cusps}} p \rfloor \geq 0\}$ for a nonzero automorphic form f of weight $2k$. That is, $S_{2k}(\Gamma) \cong L(\lfloor \operatorname{div}(f) - \sum_{\text{cusps}} p \rfloor)$. Repeat similar estimates as above (and here we need $k \geq 2$ instead of $k \geq 1$ to use the version of Riemann-Roch as above) to find that for $k \geq 2$, $\dim(S_{2k}(\Gamma)) = l(\lfloor \operatorname{div}(f) \rfloor) - \varepsilon_\infty = \dim(M_{2k}(\Gamma)) - \varepsilon_\infty$. For $k = 1$, the divisor $\lfloor \operatorname{div}(f) - \sum_{\text{cusps}} p \rfloor$ is a canonical divisor, so its linear space (which is isomorphic to $S_2(\Gamma)$) has dimension g.

We handle calculating the dimensions of the remaining spaces $M_0(\Gamma)$, $S_0(\Gamma)$ of modular forms by hand. Observe that modular forms of weight zero correspond to holomorphic functions $X(N) \to \widehat{\mathbb{C}}$; since $X(N)$ and $\widehat{\mathbb{C}}$ are compact, such maps are either surjective or constant. Since these maps have no poles they could not be surjective, so they are constants; that is, $M_0(\Gamma) \cong \mathbb{C}$ is the space of constant functions. This implies that $S_0(\Gamma) = 0$.

There exists a nonzero cusp form Δ for the congruence subgroup $\mathrm{SL}_2(\mathbb{Z})$, from which it follows Δ is a cusp form for any congruence subgroup Γ. If there were modular forms f of weight $2k$ for negative k, then $f^6\Delta$ is a weight zero cusp form; in other words, the zero function. So f had to be zero. Hence there are no modular forms of negative even weight.

We obtain the following theorem. Let k be an integer and let Γ be a congruence subgroup. Let g be the genus of $X(\Gamma)$, let $\varepsilon_2, \varepsilon_3$ be the number of elliptic points with period 2, 3 respectively, and let ε_∞ be the number of cusps. Then

$$\dim M_{2k}(\Gamma) = \begin{cases} (2k-1)(g-1) + \lfloor \frac{k}{2} \rfloor \varepsilon_2 + \lfloor \frac{2k}{3} \rfloor \varepsilon_3 + k\varepsilon_\infty & \text{if } k \geq 1, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases}$$

and

$$\dim S_{2k}(\Gamma) = \begin{cases} (2k-1)(g-1) + \lfloor \frac{k}{2} \rfloor \varepsilon_2 + \lfloor \frac{2k}{3} \rfloor \varepsilon_3 + (k-1)\varepsilon_\infty & \text{if } k \geq 2, \\ g & \text{if } k = 1, \\ 0 & \text{if } k \leq 0. \end{cases}$$