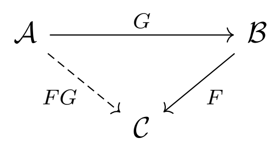
These notes closely follow *An Introduction to Homological Algebra* by Weibel.

In Abelian categories with enough injectives, there is a procedure for calculating right derived functors out of them by taking injective resolutions of objects, applying the left exact functor which is being derived, and taking cohomology. Let $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ be left exact functors of Abelian categories. We outline a procedure for calculating $R^i(FG)$ under mild conditions on the categories $\mathcal{A}$ and $\mathcal{B}$ and the functors F and G.

Theorem 0.1 (Grothendieck spectral sequence)**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ be left exact functors.*



In addition, if G sends injective objects of $\mathcal{A}$ to F-acyclic objects of B (i.e. to elements $B \in \mathcal{B}$ for which $R^iF(B) = 0$ for $i > 0$), then for each $A \in \mathcal{A}$ there exists a convergent first quadrant cohomological spectral sequence

$$E_2^{pq} = (R^pF)(R^qG)(A) \quad \textit{converging to} \quad R^{p+q}(FG)(A).$$

There are natural edge maps

$$(R^pF)(GA) \to R^p(FG)(A) \quad \textit{and} \quad R^q(FG)(A) \to F(R^qG(A))$$

and an exact sequence of low degree terms

$$0 \to (R^1F)(GA) \to R^1(FG)(A) \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)(A).$$

The main idea of the proof of Theorem 0.1 relies extensively on spectral sequences arising from two particular filtrations associated to double complexes and Cartan-Eilenberg resolutions of cochain complexes.

Given a double complex $C = C^{**}$, filter the (product or direct sum) total complex $\operatorname{Tot}(C)$ by the columns of C; that is, let ${}^IF^k \operatorname{Tot}(C)$ be given by

$$({}^IF^k \operatorname{Tot}(C))_n = \bigoplus_{\substack{p+q=n \\ p \geq k}} C^{pq}.$$

Filter the total complex $\operatorname{Tot}(C)$ by the rows of C so that ${}^{II}F^k \operatorname{Tot}(C)$ is given by

$$({}^{II}F^k \operatorname{Tot}(C))_n = \bigoplus_{\substack{p+q=n \\ q \geq k}} C^{pq}.$$

Filtering by columns yields a spectral sequence $\{{}^IE_r^{pq}\}$ with ${}^IE_0^{pq} = ({}^IF^p \operatorname{Tot}(C))_{p+q}/({}^IF^{p+1} \operatorname{Tot}(C))_{p+q} = C^{pq}$. The maps d_0 are the vertical differentials d^v of C, so ${}^IE_1^{pq} = H_v^q(C^{p*})$ with differentials d_1 given by $d_1^{pq}\colon H_v^q(C^{p*}) \to H_v^q(C^{(p+1)*})$, induced by the horizontal differentials d_h on C. Then ${}^IE_2^{pq} = H_h^pH_v^q(C)$.

Filtering by rows yields a spectral sequence $\{{}^{II}E_r^{pq}\}$ with ${}^{II}E_0^{pq} = ({}^{II}F^p \operatorname{Tot}(C))_{p+q}/({}^{II}F^{p+1} \operatorname{Tot}(C))_{p+q} = C^{qp}$. The maps d_0 are the horizontal differentials d^h of C (but are vertical in this sheet), so the maps d_1 are induced by the vertical differentials of C. Then ${}^{II}E_2^{pq} = H_v^p H_h^q(C)$.

If C is a first quadrant double complex, the filtrations ${}^{I}F^k \operatorname{Tot}(C)$ and ${}^{II}F^k \operatorname{Tot}(C)$ are canonically bounded (in this setting, this means ${}^{I,II}F^0 \operatorname{Tot}(C) = \operatorname{Tot}(C)$ and $({}^{I,II}F^{n+1} \operatorname{Tot}(C))_n = \operatorname{Tot}(C)_n$ for each n), so

$$ {}^{I}E_2^{pq} = H_h^p H_v^q(C) \quad \text{and} \quad {}^{II}E_2^{pq} = H_v^p H_h^q(C) \quad \text{converge to} \quad H^{p+q}(\operatorname{Tot}(C)). $$

This procedure is nice because it allows us to calculate homology in two different ways. We will use this technology on a Cartan-Eilenberg resolution of a particular cochain complex (yielding a double complex), to obtain the Grothendieck spectral sequence in Theorem 0.1.

Let $\mathcal{A}$ be an Abelian category with enough injectives. A (right) Cartan-Eilenberg resolution of a cochain complex A^* in $\mathcal{A}$ is an upper half-plane complex $I = I^{**}$ of injective objects of $\mathcal{A}$ and a chain map $A^* \xrightarrow{\epsilon} I^{*0}$ (called the augmentation map) such that

1. the columns I^{p*} are injective resolutions of A^p,
2. the induced maps on coboundaries and cohomology

$$ \begin{aligned} Z^p(\epsilon)&\colon Z^p(A) \to Z^p(I, d_h), \\ B^p(\epsilon)&\colon B^p(A) \to B^p(I, d_h), \\ H^p(\epsilon)&\colon H^p(A) \to H^p(I, d_h) \end{aligned} $$

   form injective resolutions, where $B^p(I, d_h)$ is the cochain complex given by $(B^p(I, d_h))^q = \operatorname{im}(d_h^{(p-1)q})$. The cochain complexes $Z^p(I, d_h)$ and $H^p(I, d_h) = Z^p(I, d_h)/B^p(I, d_h)$ are defined similarly.

Every cochain complex $A = A^*$ in $\mathcal{A}$ has a Cartan-Eilenberg resolution $A \to I$. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. We may define the (right) hyper-derived functor $\mathbb{R}^i F$ by $\mathbb{R}^i F(A) = H^i \operatorname{Tot}^\Pi(F(I))$ whenever $\operatorname{Tot}^\Pi(F(I))$ exists in $\mathcal{B}$. In the previous discussion on spectral sequences associated to filtrations of cochain complexes, let $A^p = 0$ for $p < 0$ and take C to be $F(I)$ to see that there are two spectral sequences converging to $\mathbb{R}^i F(A)$.

Proof of Theorem 0.1. Let $A \in \mathcal{A}$ and choose an injective resolution $A \to I$ of A in $\mathcal{A}$. Apply G to obtain a cochain complex $G(I)$ in $\mathcal{B}$. Then consider a Cartan-Eilenberg resolution J of $G(I)$. Since $G(I)$ has no terms in degrees less than 0, J, and hence also $F(J)$, is a first quadrant double complex so that $\operatorname{Tot}^\Pi(F(J))$ exists in $\mathcal{C}$. Thus we may consider $\mathbb{R}^i F(G(I)) = H^i \operatorname{Tot}^\Pi(F(J))$, and observe that there are two spectral sequences converging to $\mathbb{R}^i F(G(I))$.

The first spectral sequence converging to $\mathbb{R}^{p+q} F(G(I))$ is given by ${}^{I}E_2^{pq} = H_h^p H_v^q(F(J)) = H_h^p(R^q F[G(I)])$. But G takes injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$, so $R^q F[G(I^p)] = 0$ for $q > 0$. Then this spectral sequence collapses, leaving only the terms on the horizontal axis ${}^{I}E_2^{p0} = H_h^p(FG(I)) = R^p(FG)(A)$, which are the cohomology groups $H^p \operatorname{Tot}^{\Pi}(F(J)) = \mathbb{R}^p F(G(I))$.

On the other hand, the other spectral sequence converging to $\mathbb{R}^{p+q}F(G(I))$ is given by

$$ {}^{II}E_2^{pq} = H_v^p H_h^q(F(J)) \overset{☺}{=} H_v^p(F(H_h^q J)) = (R^pF)(H^q(GI)) = (R^pF)(R^qG)(A) $$

The equality (☺) involving commuting F with taking cohomology requires some elaboration. Since $J = J^{**}$ is a double complex, we obtain the short exact sequences

$$ 0 \to Z^q(J, d_h)^* \to J^{q*} \to B^{q+1}(J, d_h)^* \to 0 \quad \text{and} \quad 0 \to B^q(J, d_h)^* \to Z^q(J, d_h)^* \to H^q(J, d_h)^* \to 0. $$

Since $Z^q(J, d_h)^*$ and $B^q(J, d_h)^*$ are injective, the above short exact sequences split so that $J^{q*} = Z^q(J, d_h)^* \oplus B^{q+1}(J, d_h)^*$ and $Z^q(J, d_h)^* = B^q(J, d_h)^* \oplus H^q(J, d_h)^*$. Since F is an additive functor, applying F to the above short exact sequences preserves exactness. Thus $H_h^q(F(J)) = FZ^q(J, d_h)^*/FB^q(J, d_h)^* = (FB^q(J, d_h)^* \oplus FH^q(J, d_h)^*)/FB^q(J, d_h)^* = FH^q(J, d_h)^*$ as needed.

Thus for any $A \in \mathcal{A}$, the spectral sequence $E_2^{pq} = (R^pF)(R^qG)(A)$ converges to $\mathbb{R}^{p+q}F(G(I)) = R^{p+q}(FG)(A)$. The edge maps are indeed the natural maps $(R^pF)(GA) = E_2^{p0} \to E_\infty^{p0} \subset H^p \operatorname{Tot}^{\Pi}(F(J)) = R^p(FG)(A)$ and $R^q(FG)(A) = H^q \operatorname{Tot}^{\Pi}(F(J)) \to E_\infty^{0q} \subset E_2^{0q} = F(R^qG(A))$.

A first quadrant spectral sequence $\{E_r^{pq}\}_{r \geq 2}$ calculating the cohomology of a total complex T has $E_2^{10} = E_\infty^{10}$, since the differentials mapping to E_r^{10} have zero modules as domain and differentials with E_r^{10} as domain map to zero modules for $r \geq 2$. Then one edge map gives the first inclusion $E_2^{10} \subset H^1(T)$. Since $F^1H^1(T) \cong E_\infty^{10} = E_2^{10}$, the map $H^1(T) \to H^1(T)/F^1H^1(T) \cong E_\infty^{01} \subset E_2^{01}$ has kernel $F^1H^1(T) \cong E_2^{10}$. The differentials mapping to E_r^{01} have zero modules as domain but only the differential d_2^{01} maps to E_2^{20}, so the image of $H^1(T) \to H^1(T)/F^1H^1(T) \cong E_\infty^{01} \subset E_2^{01}$ is $\ker d_2^{10}$. Since $E_\infty^{20} = E_2^{20}/\operatorname{im} d_2^{10}$ is a quotient of E_2^{20}, the kernel of the edge map $E_2^{20} \to H^2(T)$ is $\operatorname{im} d_2^{10}$. Combining everything together, we obtain the exact sequence

$$ 0 \to E_2^{10} \to H^1(T) \to E_2^{01} \xrightarrow{d_2} E_2^{20} \to H^2(T). $$

Specializing this exact sequence to the Grothendieck spectral sequence yields the exact sequence of low degree terms $0 \to (R^1F)(GA) \to R^1(FG)(A) \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)(A)$. □

An application of the Grothendieck spectral sequence in geometry is in sheaf cohomology. Let $f\colon X \to Y$ be a continuous map of topological spaces. Consider the pushforward functor f_* on sheaves on X, which is right adjoint to the pullback functor f^{-1} on sheaves on Y. Then f_* is left exact. Because f^{-1} is exact, f_* preserves injectives. The category of sheaves on a topological space has enough injectives, so we are lead to consider composing f_* with the global sections functor Γ on sheaves on Y and applying the Grothendieck spectral sequence. The derived functors $R^i\Gamma$ form sheaf cohomology, which is usually written as $H^i(Y, -)$. For a sheaf $\mathcal{F}$ on X, observe that $\Gamma(f_*\mathcal{F}) = (f_*\mathcal{F})(Y) = \mathcal{F}(f^{-1}Y) = \mathcal{F}(X) = \Gamma(\mathcal{F})$, which we suggestively summarize as $\Gamma f_* = \Gamma$, where the second

Γ refers to taking global sections of sheaves on X. Therefore for any sheaf $\mathcal{F}$ on X, the Grothendieck spectral sequence is $E_2^{pq} = H^p(Y, R^q f_* \mathcal{F})$, which converges to $H^{p+q}(X, \mathcal{F})$.

Pushing forward to a point is the same as taking global sections: The constant map $g\colon Y \to \{*\}$ has pushforward g_* satisfying $(g_*\mathcal{G})(\{*\}) = \mathcal{G}(g^{-1}\{*\}) = G(Y) = \Gamma(\mathcal{G})$. So in general we might consider the composition of two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ and the spectral sequence $E_2^{pq} = R^p g_*(R^q f_* \mathcal{F})$ converging to $R^{p+q}(g_* f_*)\mathcal{F} = (R^{p+q}(gf)_*)\mathcal{F}$ for any sheaf $\mathcal{F}$ on X. These spectral sequences find much use in algebraic geometry.