We state the Beilinson-Bernstein localization theorem and provide a couple examples.

These notes follow *$\mathcal{D}$-Modules, Perverse Sheaves, and Representation Theory* by Hotta, Takeuchi, and Tanisaki, *Localization and Representation Theory of Reductive Lie Groups* by Miličić, *Representations of Reductive Lie Groups* (not yet published) by Mason-Brown, *Localisation de $\mathfrak{g}$-modules* by Beilinson and Bernstein, and summarize examples from *Four examples of Beilinson-Bernstein localization* by Romanov.

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1 Statement of theorem

Theorem 1.1. *Let G be a connected reductive group over an algebraically closed field k of characteristic 0, with $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Fix $\mathfrak{h}$ a Cartan subalgebra of g and a Borel subgroup B of G. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Denote by X the flag variety G/B of G.*

For any dominant and regular $\lambda \in \mathfrak{h}^*$ (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Delta$, where Δ is the root system of $\mathfrak{g}$.), there is an equivalence of categories

$$\frac{U(\mathfrak{g})}{\ker \chi_\lambda}\text{-modules} \underset{\Gamma(-)}{\overset{\mathcal{D}_\lambda \otimes_{U(\mathfrak{g})/\ker \chi_\lambda}(-)}{\rightleftarrows}} \text{quasicoherent } \mathcal{D}_\lambda\text{-modules on } X,$$

where by quasicoherent we mean quasicoherent as an $\mathcal{O}_X$-module. The map $\chi_\lambda \colon Z(\mathfrak{g}) \to k$ is the central/infinitesimal character corresponding to λ, and D_λ is a sheaf of twisted differential operators corresponding to λ.

The functor $\mathcal{D}_\lambda \otimes_{U(\mathfrak{g})/\ker \chi_\lambda} (-)$ is called localization. To elaborate on the last sentence of the theorem statement: The way the central character χ_λ is obtained from $\lambda \in \mathfrak{h}^*$ is by the Harish-Chandra isomorphism; specifically, if $\mathrm{HC} \colon Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})^W$ is the Harish-Chandra isomorphism (with W the Weyl group obtained from $\mathfrak{h}$), then $\chi_\lambda(z) = \lambda(\mathrm{HC}(z))$. Because λ is linear and k is a commutative algebra, λ determines a unique map of algebras $\mathrm{Sym}(\mathfrak{h}) \to k$, and restriction to $\mathrm{Sym}(\mathfrak{h})^W$ gives a map which is unique up to the Weyl group orbit of λ. That is, any element in the Weyl group orbit of λ determines the same infinitesimal character. The trivial infinitesimal character is obtained from any element in the Weyl group orbit of $\lambda = \rho$, the half-sum of the positive roots in Δ.

We will discuss $\mathcal{D}_\lambda$-modules in the next section, after some exposition on sheaves of twisted differential operators.

2 Sheaves of twisted differential operators

2.1 Definitions

The ordinary sheaf of differential operators $\mathcal{D}_X$ on X may be obtained in a few ways; one such way is by considering the sheaf generated by $\mathcal{O}_X$ and the tangent sheaf $\mathcal{T}_X$, both viewed as subsheaves of $\mathcal{E}nd_{k_X}(\mathcal{O}_X)$ (where $\mathcal{O}_X$ includes into $\mathcal{E}nd_{k_X}(\mathcal{O}_X)$ as multiplication by functions and we view the tangent sheaf as the sheaf of k_X-linear derivations of $\mathcal{O}_X$). We consider sheaves of "twisted differential operators", which slightly generalize the ordinary sheaf of differential operators.

Consider the category of pairs $(\mathcal{A}, f)$, where $\mathcal{A}$ is a sheaf of k-algebras on X and $f\colon \mathcal{O}_X \to \mathcal{A}$ is a morphism of sheaves of k-algebras. There is the natural inclusion $i\colon \mathcal{O}_X \to \mathcal{D}_X$, so it follows that the pair $(\mathcal{D}_X, i)$ belongs to the aforementioned category.

Definition 2.1. The pair $(\mathcal{A}, f)$ (often writing $(\mathcal{A}, f)$ as $\mathcal{A}$) is called a sheaf of *twisted differential operators* if $(\mathcal{A}, f)$ is locally isomorphic to the pair $(\mathcal{D}_X, i)$. †

A more concrete description: Let $(\mathcal{A}, f)$ be a sheaf of twisted differential operators. Then there exists an open cover $\mathcal{U} = \{U_j \mid 1 \le j \le n\}$ of X for which $(\mathcal{A}, f)|_{U_j}$ is isomorphic to $(\mathcal{D}_{U_j}, i_{U_j})$, for each $1 \le j \le n$. For each $1 \le j \ne k \le n$, there is also an automorphism ϕ_{jk} of $(\mathcal{D}_{U_j \cap U_k}, i_{U_j \cap U_k})$ for which the diagram

$$\begin{array}{ccc} & (\mathcal{A}, f)|_{U_j \cap U_k} & \\ \overset{\cong}{\swarrow} & & \overset{\cong}{\searrow} \\ (\mathcal{D}_{U_j \cap U_k}, i_{U_j \cap U_k}) & \xrightarrow[\phi_{jk}]{} & (\mathcal{D}_{U_j \cap U_k}, i_{U_j \cap U_k}) \end{array}$$

commutes.

From a line bundle L over X, we can construct a sheaf of twisted differential operators using the following recipe. Consider the corresponding locally free $\mathcal{O}_X$-module $\mathcal{L}$ of sections of L. We define the sheaf $\mathcal{D}_X^{\mathcal{L}}$ by the filtration $\mathcal{D}_X^{\mathcal{L}} = \bigcup_{n \in \mathbb{Z}} F_n \mathcal{D}_X^{\mathcal{L}}$, where

$$F_n \mathcal{D}_X^{\mathcal{L}} = \begin{cases} 0 & n < 0, \\ \{p \in \mathcal{E}nd_{k_X}(\mathcal{L}) \mid [p, f] \in F_{n-1}\mathcal{D}_X^{\mathcal{L}} \text{ for } f \in \mathcal{O}_X \subset \mathcal{E}nd_{k_X}(\mathcal{L})\} & n \ge 0. \end{cases}$$

Note that $F_0 \mathcal{D}_X^{\mathcal{L}} = \mathcal{O}_X \subset \mathcal{E}nd_{k_X}(\mathcal{L})$, and that $\mathcal{L} = \mathcal{O}_X$ recovers the ordinary sheaf of differential operators; that is, $\mathcal{D}_X = \mathcal{D}_X^{\mathcal{O}_X}$.

The descriptions above tell us that by taking a sheaf of functions $\mathcal{L}$ given by regular functions on open sets covering X glued together in some way, we can obtain a sheaf of twisted differential operators $\mathcal{D}$ by taking ordinary sheaves of differential operators on these open sets and gluing them together in a manner compatible with the gluing of the functions in $\mathcal{L}$.

2.2 Equivariant sheaves

Let G act on X. If $\mathcal{L}$ is a G-equivariant sheaf, then there is a ring homomorphism $\varphi\colon U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^{\mathcal{L}})$. In the case that $k = \mathbb{C}$, we can describe ϕ by the familiar formula

$$\phi(a)s = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp(ta)s(\exp(ta)^{-1}(-))\Big)\Big|_{t=0}$$

for $a \in \mathfrak{g}$ and $s \in \mathcal{L}$.

Since $\Gamma(X, \mathcal{L})$ is a representation of G, it is also a $U(\mathfrak{g})$-module, where the map $U(\mathfrak{g}) \to \mathrm{End}(\Gamma(X, \mathcal{L}))$ coincides with the composition of the maps $U(\mathfrak{g}) \xrightarrow{\varphi} \Gamma(X, \mathcal{D}_X^{\mathcal{L}}) \subset \mathrm{End}(\Gamma(X, \mathcal{L}))$.

We are interested in forming these equivariant sheaves in the case when X is the flag variety G/B, in order to define the sheaves of twisted differential operators D_λ. We will restrict ourselves to the case when λ is an element of the weight lattice of the root system of $\mathfrak{g}$; that is, an element of $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$.

Let $\lambda \in P$. Then the element $\lambda - \rho$ determines a character/one-dimensional representation of B, which we denote by $\exp(\lambda - \rho)$ (by extending $\lambda - \rho$ to a map on $\mathfrak{b}$ and exponentiating; maybe we need G to be simply-connected for this to work on the nose?). Form the associated G-equivariant line bundle $L = G \times^B \exp(\lambda - \rho)$ and take its sheaf of sections $\mathcal{L}$. Then $\mathcal{D}_\lambda$ is defined to be the sheaf of twisted differential operators $\mathcal{D}_X^{\mathcal{L}}$. The ρ-shift here is needed in order to match the convention that the trivial infinitesimal character corresponds to the Weyl group orbit of $\lambda = \rho$.

It follows that global sections of $\mathcal{D}_\lambda$-modules are $U(\mathfrak{g})$-modules. The content of the Beilinson-Bernstein localization theorem is to add adjectives to λ, $\mathcal{D}_\lambda$, and $U(\mathfrak{g})$-modules until the global sections and localization functors define an equivalence of categories (and prove that they do so). In these notes we restrict our view to special cases for sake of being able to work out examples, which we do in the following section.

3 Examples for $\mathrm{SL}_2(\mathbb{C})$

What follows will be heavily abridged calculations, which we hope are reproducible given the scaffolding below. The details are a combination of my calculations and Romanov's. Let $k = \mathbb{C}$ and $G = \mathrm{SL}_2(\mathbb{C})$. Give $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ the standard basis $\{e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$ and choose the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$. Note that the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ is the polynomial algebra $\mathbb{C}[c]$, where $c = h^2/2 + fe + ef$ is the Casimir element for $\mathfrak{sl}_2(\mathbb{C})$. Choosing B to the be subgroup of upper triangular matrices, we can identify $X = G/B$ with $\mathbb{P}^1$ via $gB \mapsto g[1:0]$, where $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)[u:v] = [au + bv : cu + dv]$.

Give $\mathbb{P}^1$ the usual open cover $\{U_0, U_\infty\}$, where $U_0 = \mathbb{P}^1 \setminus \{\infty = [0:1]\}$ and $U_\infty = \mathbb{P}^1 \setminus \{0 = [1:0]\}$. Both open sets are identified with copies of $\mathbb{C}$, U_0 with coordinate z and U_∞ with coordinate w. The transition function is the inversion map $z \mapsto 1/z$. In view of the identification $G/B \cong \mathbb{P}^1$, the open set U_0 corresponds to $\{\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\}B$ and U_∞

corresponds to $\{\left(\begin{smallmatrix} w & -1 \\ 1 & 0 \end{smallmatrix}\right)\}B$. The transition function between these two subsets of G/B is the identity map since for $z \neq 0$, $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/z & -1 \\ & z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1/z & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Both descriptions will be useful.

The root system for $\mathfrak{sl}_2(\mathbb{C})$ consists of two elements, $\{-2, 2\}$, where by ± 2 we mean the maps $h \mapsto \pm 2$. It follows that ρ is the map $1\colon h \mapsto 1$, and that the weight lattice is $\mathbb{Z}\rho$.

The character/one-dimensional representation of B obtained from the weight $n\rho - \rho$ is the map $\exp(n\rho - \rho)\colon \left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right) \mapsto a^{n-1}$. Sections of the associated bundle $G \times^B \exp(n\rho - \rho)$ on the open set $\{\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\}B \cap \{\left(\begin{smallmatrix} w & -1 \\ 1 & 0 \end{smallmatrix}\right)\}B$ look like

$$\begin{aligned} s\colon \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)B \mapsto \left(\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right), f\left(\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\right)\right) &= \left(\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/z & -1 \\ & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 1/z & -1 \\ & z \end{smallmatrix}\right)^{-1} f\left(\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/z & -1 \\ & z \end{smallmatrix}\right)\right)\right) \\ &= \left(\left(\begin{smallmatrix} 1/z & 1 \\ -1 & 0 \end{smallmatrix}\right), z^{n-1} f\left(\left(\begin{smallmatrix} 1/z & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\right), \end{aligned}$$

and this back-of-envelope calculation justifies identifying the sheaf of regular sections of $G \times^B \exp(n\rho - \rho)$ with the Serre twisting sheaf $\mathcal{O}(n-1)$.

The sheaf of twisted differential operators $\mathcal{D}_{n\rho} = \mathcal{D}_X^{\mathcal{O}(n-1)}$ is given explicitly by defining it to be $\mathcal{D}_{U_i}$ on U_i for $i = 0, \infty$, and gluing on $U_0 \cap U_\infty$ by the map

$$\psi\colon z \mapsto z, \quad \partial_z \mapsto \partial_z - \frac{n-1}{z} = z^{n-1}\partial_z z^{-(n-1)}.$$

(That is, the differential operator on the right hand side untwists, differentiates, and then retwists).

Since $\mathcal{O}(n-1)$ is a G-equivariant sheaf (by translation), we can realize $e, h, f \in \mathfrak{g}$ as the following differential operators on U_0, U_∞ (using z as the coordinate on U_0 and w as the coordinate on U_∞):

$$\begin{aligned} e &= z^2\partial_z - z(n-1) = -\partial_w, \\ h &= 2z\partial_z - (n-1) \ = -2w\partial_w + (n-1), \text{ and} \\ f &= -\partial_z \qquad\qquad\ \ = w^2\partial_w - w(n-1). \end{aligned}$$

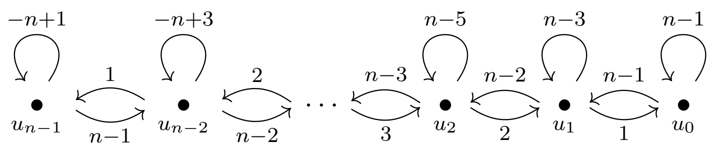
One can check that these expressions are compatible with the ring map ψ above, meaning that to go from a differential operator in one coordinate to the other, we apply ψ. Therefore, e, h, f may be realized as global sections of $\mathcal{D}_{n\rho}$.

3.1 Recovering the Borel-Weil theorem

Consider $\mathcal{O}(n-1)$ itself as a $\mathcal{D}_{n\rho}$-module for $n \geq 0$. Global sections of $\mathcal{O}(n-1)$ may be identified with the space of homogeneous polynomials in two variables of degree $n-1$. Alternatively, by looking only at the sections on say the U_∞ open set, we can identify global sections of $\mathcal{O}(n-1)$ with polynomials in w of degree less than or equal to $n-1$. A convenient basis for this vector space is $\{u_k\}_{1 \leq k \leq n-1} = \{1, -w, \ldots, (-1)^k w^k, \ldots, (-1)^{n-1} w^{n-1}\}$; and we calculate the action of the differential operators e, h, f on these basis elements (using the expressions on the right hand side from above):

$$\begin{aligned} eu_k &= ku_{k-1}, \\ hu_k &= ((n-1)-2k)u_k, \text{ and} \\ fu_k &= ((n-1)-k)u_{k+1}. \end{aligned}$$

A pictorial representation of these actions is given by nodes representing the $\mathbb{C}$-spans of each basis element u_k and arrows indicating how each differential operator takes the basis element to a scalar multiple of some other basis element.



Here e acts as a lowering operator, f as a raising operator, and h as a diagonal matrix (this is reversed from the usual roles of e and f since we chose to look at the sections on U_∞ and not U_0). The corresponding infinitesimal character can be calculated by explicitly finding out how the Casimir element $c = \frac{1}{2}h^2 + fe + ef$ acts on u_k, and it will end up being $(n^2-1)/2$. So we have recovered the irreducible n-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ of highest weight $n-1$, which matches the result from the usual Borel-Weil theorem.

3.2 Recovering Verma modules

This example will require some blackboxing of the theory of $\mathcal{D}$-modules and it would probably be harder to reproduce the result below. Consider the restriction of the action of G on $\mathbb{P}^1$ to the unipotent subgroup $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\}$ of G. The orbits of this action are $\{0\} = 0$ and U_∞.

We will consider just the point orbit (considering the other orbit is a dual situation to this one). The structure sheaf on 0 may be pushed forward along the inclusion $i_0 \colon 0 \to X$ to a $\mathcal{D}_{n\rho}$-module $i_{0+}\mathcal{O}_0$ on X. Since 0 is affine and the support of this pushforward lies entirely in the affine open U_0, we only need to look at the $\mathcal{D}_{U_0}$-module structure of the direct image of $\mathcal{O}_0$.

This direct image is given by $\mathcal{D}_{U_0 \leftarrow 0} \otimes_{\Gamma(0,\mathcal{D}_0)} \Gamma(0,\mathcal{O}_0) = \mathcal{D}_{U_0 \leftarrow 0} \otimes_{\mathbb{C}} \mathbb{C} = \mathcal{D}_{U_0 \leftarrow 0}$, where $\mathcal{D}_{U_0 \leftarrow 0} = \Gamma(0,\mathcal{O}_0) \otimes_{\Gamma(U_0,\mathcal{O}_{U_0})} \Gamma(U_0,\mathcal{D}_{U_0})$ is the left $\Gamma(U_0,\mathcal{D}_{U_0})$-module where the module structure is given by multiplication on the right factor by the transpose/formal adjoint of a differential operator. Since $\Gamma(0,\mathcal{O}_0) \cong \mathbb{C}[z]/(z)$ as $\Gamma(U_0,\mathcal{O}_{U_0}) \cong \mathbb{C}[z]$-modules, the direct image of the structure sheaf on 0 is isomorphic to the $\Gamma(U_0,\mathcal{D}_{U_0}) \cong \mathbb{C}\langle z, \partial_z\rangle/((\partial_z z - z\partial_z - 1))$-module

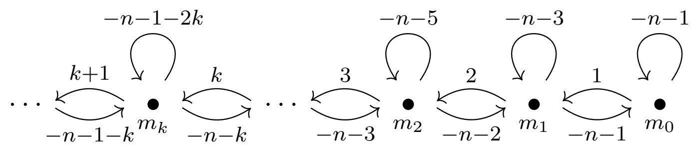
$$\Gamma(U_0,\mathcal{D}_{U_0})/\Gamma(U_0,\mathcal{D}_{U_0})z = \bigoplus_{i=0}^{\infty} \partial_z^i \delta,$$

where $\delta \colon \mathbb{C} \to \mathbb{C}$ is the indicator function of 0 (a Kronecker delta as opposed to a Dirac delta since we are working with left $\mathcal{D}$-modules $\approx$ spaces of functions, as opposed to right $\mathcal{D}$-modules $\approx$ spaces of distributions). Nevertheless these derivatives are taken formally or perhaps in the sense of distributions, and we can calculate the action of e, h, f on this space.

Choose the convenient basis $\{m_k = (-1)^k \partial_z^k \delta / k!\}_{k \geq 0}$ for the direct image module above. Then some direct calculations give

$$\begin{aligned} em_k &= (-n-k)m_{k-1} \quad \text{for } k > 0, \\ em_0 &= 0, \\ hm_k &= (-n-1-2k)m_k, \text{ and} \\ fm_k &= (k+1)m_{k+1}. \end{aligned}$$

The picture in this case looks like



Here e acts as a raising operator, f as a lowering operator, and h semisimply. The corresponding infinitesimal character we calculate by calculating how the Casimir element acts on m_k, which in this case is also by $(n^2-1)/2$. So this procedure recovers the irreducible Verma module of highest weight $-n-1$.